Optical Methods in Experimental Mechanics

Part 48: Measuring Phase Difference—Part IV: Phase-Stepping Algorithms

REVIEW AND PURPOSE

The previous three articles in this series laid the groundwork for completely mapping interferometric phase difference through the use of only intensity measurements. Part 45 formulated the problem for ideal and non-ideal interference and described the ambiguities that must be faced. Part 46 examined the compensation technique, which is the forerunner of computer-based phase mapping. Part 47 described a simple interferometric system that incorporates a phase shifter, intensity detectors, and a computer.

This article and the next one explain how to perform whole-field phase-stepping interferometry by showing what measurements are needed and what is done with them. The equations required for two of several possible methods of data reduction as well as a metric for judging data validity are derived.

SUMMARY OF PROCEDURE

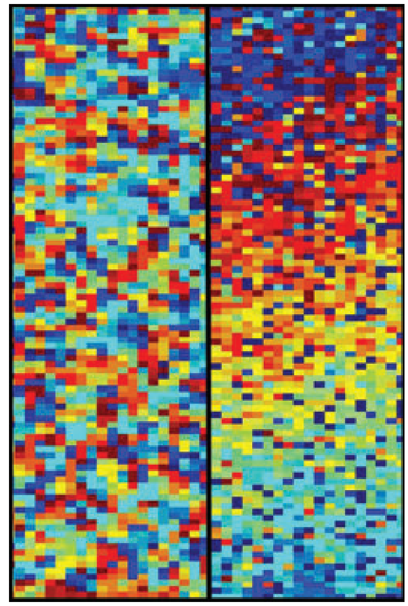
Description of the data acquisition and reduction processes for phase-shifting interferometery will be spread over two articles. Confusion might be diminished and continuity cemented if the steps required are enumerated up front. Determination of the phase difference profile over the optical field necessitates the following steps:

1. record and store an intensity map for each of three or more phase steps,
2. perform the required phase difference calculations for each detector (pixel) in the field,
3. eliminate ambiguities and adjust the calculated phase difference to modulo 2π,
4. remove the 2π limitation to obtain the final correct phase difference,
5. store and display the results.

Steps 3 and 4 of the list will be described in the next article. Already, you will perceive that mapping phase difference requires significant data acquisition and computational capability. You can do it by hand for one detector or a few, of course, as described in Part 46. Otherwise, as mentioned in Part 47, the process is automated by exploiting an electronic camera and a computer. By modern standards, only minimal computer power is enough.

A NOTATION CHANGE

Now that the distinctions between absolute phase, relative phase or phase difference (the difference between optical path lengths in radian units that we



***Left*: Enlarged portion of a false-color phase difference map obtained from phase-stepping speckle interferometry applied to a beam. The phase difference varies randomly between speckles (pixels). *Right*: Corresponding map of the change of phase difference obtained by subtracting the initial phase difference from the final, pixel-by-pixel. The phase difference changed by one wavelength between the upper and lower blue regions as the beam was loaded. These results have not been filtered or smoothed. Digital maps courtesy of Dr. Gaetano Restivo of Michigan State University, 2011.**

This article:

- *describes the steps to perform whole-field phase-stepping interferometry,*
- *develops the equations for two methods of data reduction.*

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- *record intensity maps for three or more phase steps,*
- *perform phase difference calculations for each detector in the array,*
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- *remove the 2π limitation to obtain the final correct phase difference,*
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The series, Optical Methods—Back to Basics, *is written by University Distinguished Professor Gary Cloud (SEM Fellow) of Michigan State University in East Lansing, Michigan. It began by introducing the nature and description of light and is progressing through topics ranging from diffraction through phase shifting interferometries. The intent is to educate through narrative discussion and marginal summaries coupled with illustrative photos and diagrams that can be used by practitioners in the classroom as well as in industry. Professor Cloud is internationally known for his work in optical measurements and for his book,* Optical Methods of Engineering Analysis. *Unless otherwise noted, the graphics in this series were created by the author.*

If you have comments or questions about this series, please contact Jen Tingets, journals@sem1.com.

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have called $\delta\phi$), and the change of phase difference (change of path length difference) are understood, the mathematical notation that was used earlier can be simplified. From this point onward, ϕ is used to represent the phase difference between the interfering waves at a point in the detector array. $\Delta\phi$ will be used later to identify the **change** of phase difference, which is the quantity usually needed to calculate the change of specimen state (e.g. displacement) that occurs during an experiment.

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STARTING POINT AND DEFINITIONS

Methods of calculating phase difference at each detector location from intensity measurements can begin with any one of equations 45.7 through 45.11, as derived in Part 45. For convenience, and to match the forms usually given in research papers, begin with the expression in eqn. 45.11, reproduced here with the notation changed as mentioned above,

$$I_s = \frac{I_{\max} + I_{\min}}{2} + \frac{I_{\max} - I_{\min}}{2} \cos\phi \tag{48.1}$$

Drop the subscript on the left-hand side and remember that I is the total intensity at a particular detector location. Additional physical insight and utility are gained if this equation for the total intensity at the detector is modified to the following form,

$$I = \frac{I_{\max} + I_{\min}}{2} \left[1 + \left(\frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}\right) \cos\phi\right] \tag{48.2}$$

The first expression on the right is the average intensity, as explained earlier, so call it I_{av} from now on. The multiplier of $\cos\phi$ inside the brackets is the ratio of the intensity oscillation to the average intensity. It seems to be a signal-to-noise figure that tells us the degree to which the intensity variations differ from the background intensity. When observing interference fringes by eye or when photographing them, this parameter tells us whether or not the fringes can be clearly seen, so it is called "fringe visibility." In electronic detection or in communications applications, this quantity is the "intensity modulation," and we will label it I_v. Calculation of the modulation in an experiment is useful, indeed necessary, in order to decide if a measured phase difference is valid. With these changes in place, the output of the detector at coordinates (x, y) in the detector array takes the form,

$$I(x, y) = I_{av}(x, y)[1 + I_v(x, y)\cos\phi(x, y)] \tag{48.3}$$

The intensity at a detector = average intensity(1 + modulation times cosine of the phase difference).

In the following development of algorithms, the coordinate specifier (x, y) will be dropped for brevity. Please do not forget that the calculations must be performed for each and every detector in the array in order to map the phase difference over the field.

As has been pointed out several times, any form of the intensity-phase relationship contains three unknowns, so at least three intensity observations are required to obtain the desired phase difference ϕ at each detector. In equation 48.3, the unknowns are I_{av}, I_v, and ϕ, the latter two being of most interest.

The unknowns are:

- *the average intensity,*
- *the intensity modulation,*
- *the phase difference.*

APPROACH AND ASSUMPTIONS

Many approaches and algorithms have been implemented for determining phase difference using intensity measurements and phase shifting or phase stepping. Only two of the most basic and most useful are described here. At the start, attention is confined to discrete phase stepping, meaning that known increments of phase difference are imposed via the phase shifter. After each step, the shifter device completely stops so that an intensity map for that step can be recorded.

This procedure requires that the phase shifter is calibrated in advance, or else some means of measuring the phase shifts on the fly must be provided. The need to stop and start the phase shifter tends to slow the data acquisition, but the discrete stepping approach is the easiest of the alternatives to understand. It gives quality results, and the phase stepping, intensity recording, and calculations can even be done by hand in elementary experiments that demonstrate validity and utility. A more sophisticated method, to be described in a later article, averages the intensities while phase shifts are occurring. These techniques are quicker, because phase shifter motion is continuous. We will find that the data processing algorithms are identical to those discussed below for phase stepping, except that the form of the modulation function is slightly more complicated.

Assume that:
- *the phase shifter is calibrated,*
- *the phase shifts are applied in known discrete steps (phase stepping),*
- *the phase shifter is stopped at each step while intensity data are recorded.*

For the time being, sampling requirements are ignored. Assume that sufficient detectors are in place and the detectors are small enough so that the intensity variations over the field can be accurately mapped. Also, assume that the time resolution is good enough to accommodate any important dynamic phenomena in the experiment.

THREE-STEP TECHNIQUE

An algorithm using three discrete phase steps demonstrates that the phase difference can, indeed, be obtained with only three measurements. The technique is easy to implement, and it works very well for most applications.

Start with equation 48.3, but, somewhat oddly, we find that the math is simpler if we do not use the intensity at zero phase shift. Instead, take intensity readings from the array of detectors for total phase steps of $\pi/4$, $3\pi/4$, and $5\pi/4$. The intensities recorded for this series of phase steps will be,

In the three-step technique, intensity maps are recorded for phase steps of $\pi/4$, $3\pi/4$, and $5\pi/4$. I_1, I_2, and I_3 are the intensities recorded at any specific detector for these three phase steps.

$$
\begin{aligned}
I_1 &= I_{av}\left[1 + I_v \cos\left(\phi + \frac{\pi}{4}\right)\right] \\
I_2 &= I_{av}\left[1 + I_v \cos\left(\phi + \frac{3\pi}{4}\right)\right] \\
I_3 &= I_{av}\left[1 + I_v \cos\left(\phi + \frac{5\pi}{4}\right)\right]
\end{aligned}
\tag{48.4}
$$

Use the identity for the cosine of the sum of two angles, evaluate the sine and cosine of the phase step terms, then tidy up to obtain,

$$
\begin{aligned}
I_1 &= I_{av}\left[1 + \frac{\sqrt{2}}{2} I_v(+\cos\phi - \sin\phi)\right] \\
I_2 &= I_{av}\left[1 + \frac{\sqrt{2}}{2} I_v(-\cos\phi - \sin\phi)\right] \\
I_3 &= I_{av}\left[1 + \frac{\sqrt{2}}{2} I_v(-\cos\phi + \sin\phi)\right]
\end{aligned}
\tag{48.5}
$$

Now, calculate the following differences,

$$
\begin{aligned}
I_1 - I_2 &= \sqrt{2} I_{av} I_v \cos\phi \\
I_3 - I_2 &= \sqrt{2} I_{av} I_v \sin\phi
\end{aligned}
\tag{48.6}
$$

Divide equals by equals and use the common identity for tangent to obtain,

$$
\phi = \arctan\left[\frac{\sin\phi}{\cos\phi}\right] = \arctan\left[\frac{I_3 - I_2}{I_1 - I_2}\right] \tag{48.7}
$$

At each detector, the phase difference is found to be the arctangent of the ratio $(I_3 - I_2)/(I_1 - I_2)$.

We see that, indeed, the phase difference can be obtained from the three intensity measurements. Recall that this calculation is carried out for every detector in the array. Two problems remain. We still do not know which cycle of the intensity curve we are on, and the tangent calculation leaves us wondering which quadrant we are in. Set these ambiguities aside for awhile. We said that the modulation is needed in order to judge the quality of the measurement. Compute it by squaring equations 48.6.

The arctangent function yields the phase difference modulo π, meaning that we do not know for sure in which quadrant the correct value lies. This ambiguity must be eliminated.

$$(I_1 - I_2)^2 = 2I_{av}^2 I_v^2 \cos^2\phi \qquad (48.8)$$
$$(I_3 - I_2)^2 = 2I_{av}^2 I_v^2 \sin^2\phi$$

Add these two equations, use the obvious identity, and sort to obtain the modulation metric,

$$I_v = \frac{\sqrt{(I_1 - I_2)^2 + (I_3 - I_2)^2}}{2I_{av}} \qquad (48.9)$$

This modulation result contains the average intensity, so use parts of equations 48.5 again to find out what it is, with the result being,

$$I_{av} = \frac{I_1 + I_3}{2} \qquad (48.10)$$

This interesting result suggests that the average intensity can be obtained directly from any two measurements at phase steps that are 180° apart. You could have guessed this useful fact from examination of the graphs in Part 45. Now, combine equations 48.9 and 48.10 to obtain the modulation in terms of the specific set of intensities recorded by the detector.

The modulation metric is also obtained in terms of the intensity data.

$$I_v = \frac{\sqrt{(I_1 - I_2)^2 + (I_3 - I_2)^2}}{I_1 + I_3} \qquad (48.11)$$

In summary, so far, we have completely mapped the phase difference and obtained information about the significance of the data from three intensity maps taken at specific phase steps. Before attending to the ambiguities mentioned above, consider another stepping technique that has considerable merit.

FOUR-STEP TECHNIQUE

A technique that uses four equal phase steps has certain advantages over the three-step implementation, and it is widely used. In this case, the initial intensity at zero phase step is used along with three others at $\pi/2$ increments, meaning intensity maps are recorded at phase steps of *0*, $\pi/2$, π, and $3\pi/2$. Substitute these values into equation 48.3 to establish the starting point,

For the four-step technique, intensity maps are recorded at phase steps of 0, $\pi/2$, π, and $3\pi/2$. The recorded intensities at each detector are I_1 through I_4.

$$\begin{aligned} I_1 &= I_{av}[1 + I_v \cos\phi] \\ I_2 &= I_{av}\left[1 + I_v \cos\left(\phi + \frac{\pi}{2}\right)\right] \\ I_3 &= I_{av}[1 + I_v \cos(\phi + \pi)] \\ I_4 &= I_{av}\left[1 + I_v \cos\left(\phi + \frac{3\pi}{2}\right)\right] \end{aligned} \qquad (48.12)$$

Solution of these equations parallels that for the three-step technique except that it is actually simpler. The result for the phase difference at each detector is,

At each detector, the phase difference is found to be the arctangent of the ratio $(I_4 - I_2)/(I_1 - I_3)$.

$$\phi = \arctan\left[\frac{\sin\phi}{\cos\phi}\right] = \arctan\left[\frac{I_4 - I_2}{I_1 - I_3}\right] \qquad (48.13)$$

The average intensity in terms of the intensity readings is,

$$I_{av} = \frac{I_1 + I_3}{2} = \frac{I_2 + I_4}{2} \tag{48.14}$$

And, the modulation in terms of the intensity measurements turns out to be,

$$I_v = \frac{\sqrt{(I_1 - I_3)^2 + (I_4 - I_2)^2}}{I_1 + I_3} = \frac{\sqrt{(I_1 - I_3)^2 + (I_4 - I_2)^2}}{I_2 + I_4} \tag{48.15}$$

Notice that, because four intensity maps were recorded when only three were required, some redundancies appear in the equations for average intensity and modulation, and these are useful for crosschecking.

WHAT IS NEXT?

The final few articles in this series will show how to eliminate ambiguities inherent in the arctan function as well as establish continuity at the 2π boundaries, thus converting the phase differences from modulo π to the actual values attained in the image field. Also to be described are ways to utilize intensity measurements recorded on the fly (phase shifting instead of stepping) and the associated modifications in the equations given above. Finally, we will learn how to obtain specimen displacement from the phase difference, and, time permitting, show an improved but simple setup along with sample software. ■

The next articles will discuss, time permitting:

- *eliminating ambiguities in the arc-tangent function,*
- *establishing continuity at 2π boundaries,*
- *collecting data on the fly (phase shifting),*
- *interpreting phase difference data to determine specimen displacement,*
- *a simple setup,*
- *sample software.*