Optical Methods in Experimental Mechanics

Part 38: Photoelasticity X—Transfer of Stress Data From Model to Prototype

REVIEW AND PURPOSE

The previous several articles in this series have shown how photoelasticity can be implemented to determine the stress state in a transparent model of a prototype structural component.

The objective now is to transfer the stress data from the photoelastic model to the prototype, which likely is made of a different material, carries different loads, and is of different overall size. The effects of material properties, loads, and geometric scaling must be analyzed, and the appropriate scaling laws must be developed.

THE QUESTIONS

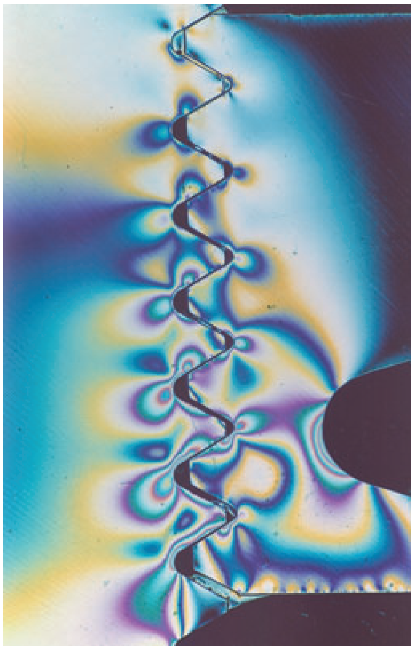
If you demonstrate photoelastic stress analysis to a group of engineers or lay people, or when you present photoelasticity results to management, the following question will be raised sooner or later, in one form or another, and maybe in a tone that implies dismissal. *"Well, you have obtained the stress distribution in a plastic model, but what does that prove about the stresses in the actual part, which is to be made of steel?"* If not steel, maybe the prototype structure is made of concrete, composite, bone, ice, cardboard, or collagen. How do you handle the differences in material properties?

The loads applied to the model are probably different from those applied to the real structure. Certainly, a plastic model cannot resist as much load as a metal prototype, even if they are of the same size. How should you account for this load difference?

A final aspect of the question has to do with the possibility, dictated by necessity or convenience, that the model be larger or smaller than the prototype. You might be studying the stresses in a river dam, which is certainly too large to model full scale in your laboratory. On the other hand, the analysis might involve stress measurement around a single miniscule fiber in a composite, in which case you might rather make the model many times larger than life size. How does one scale the results of the stress analysis to account for the different sizes of model and prototype?

The need to transfer experimental results from model to prototype raises profound questions about the dependence of stress magnitudes and distributions on material properties, load magnitudes, load directions, relative sizes of model and prototype, and a host of other factors.

These questions are not confined to the arena of experimental solid mechanics. The inventors of controlled flight, Orville and Wilbur Wright, were successful



Dark-field, white-light isochromatic fringe pattern showing the stress distribution near tapered threads in a bolted joint. Photo courtesy of Dr. Eann Patterson, Professor of Mechanical Engineering and Director of the Composite Vehicle Research Center, Michigan State University.

Stresses obtained from a photoelastic model are transferred to the prototype, which likely differs from the model in material, size, and shape.

To determine stresses in the prototype from model studies, the following questions must be answered.

- *How does one account for the difference of material properties?*
- *How does one adjust for the difference of loads?*
- *May one make the model larger or smaller than the prototype, and, if so, how does one compensate for the size difference?*

The series, Optical Methods—Back to Basics, is written by University Distinguished Professor Gary Cloud of Michigan State University in East Lansing, Michigan. It began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless otherwise noted, the graphics in this series were created by the author.

The series author, Professor Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his book, Optical Methods of Engineering Analysis.

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because they pioneered small-scale wind tunnel testing of the flow of air over model airfoils and shrewdly inferred that their measurements of lift could be applied to a full-scale airplane wing. Fluid mechanists have since solved similar problems via modeling techniques for many years. Civil engineers fabricate laboratory models of harbors, skyscrapers, and viaducts to study responses of these structures to everything from earthquakes to tsunamis. Biomechanists have employed experimental models to study tooth, hoof, horn, and feather.

Profound questions require profound answers. Indeed, the general field of model similarity and scaling is deep, sophisticated, and fascinating. But, more narrowly focused applications allow answers that are less general and less sophisticated, but still serviceable, correct, and adequate. Such is the case with basic applied photoelasticity.

The following sections present first a very simple approach to dealing with the material properties question in photoelasticity, as well as the limitations to that approach. Then, the elementary techniques to attend to size and load scales are offered. Finally, some aspects of the general theory for dealing with size and material differences between model and prototype are outlined in order to point the way for those who wish to gain fuller understanding of this complex and interesting subject.

Transfer of results from a model experiment to the prototype is:
- *a profound topic,*
- *productively applied in many fields including, among many more,*
  - *aviation,*
  - *fluid mechanics,*
  - *geomechanical engineering,*
  - *biomechanics,*
- *based on dimensional analysis and known related solutions.*

The general field of dimensional analysis is sophisticated, but specific applications facilitate less-general solutions that are correct and serviceable.

MATERIAL PROPERTIES

Go back to the question raised at the top of this article about how the material properties affect the measured stress distributions. Recall from mechanics of deformable solids or elementary elasticity the equations for stress in typical structural components, for example:

$$\sigma = \frac{P}{A} \text{ for a tension specimen} \tag{38.1}$$

$$\sigma = \frac{My}{I} \text{ for a beam in bending} \tag{38.2}$$

Even experienced engineers are sometimes startled by the realization that neither the modulus of elasticity nor Poisson's ratio appears in these relationships. Only loads and geometric properties are factors. The tentative conclusion is that material properties are not important, so stresses that are measured in a photoelastic model are identical to what they would be in a metal prototype of the same size and shape and under the same load. It seems that *material properties need not be considered when transferring stress from model to prototype.*

This inference drawn above from elementary considerations is true and can be utilized in the great majority of applied photoelasticity studies; but, of course, there are exceptions. Utilization of elasticity theory leads to the following summary statements about the importance of material properties for studies of objects made of homogeneous isotropic materials. Except where noted, it is assumed that the body forces are zero or constant and there are only traction (force) boundary conditions.

1. If the object being studied is "simply connected," meaning it has no holes, then the stress distribution is completely independent of material properties.
2. For "multiply connected shapes"—ones with holes—that are free of unequilibrated force on any boundary, the stress is again independent of properties.
3. If any boundary carries an unequilibrated load, such as usually happens with fasteners, for example, then the stress distribution depends to some degree upon Poisson's ratio but not the elastic modulus.

Basic solutions for stress in structural components do not include material properties, so we infer that material properties need not be considered when transferring stresses from model to prototype. This conclusion does not apply to all cases, so more comprehensive study is required.

Elasticity theory leads to the following rules that govern the importance of the properties of model and prototype:
- *For shapes with no holes, material properties need not be considered.*
- *For shapes with holes that are free of unequilibrated force on any boundary, the material properties need not be considered.*
- *If any boundary of a shape with holes carries an unequilibrated load, then Poisson's ratio is a small factor in the stress distribution.*
- *If the body forces are not zero or constant, then the Poisson ratio is a small factor.*
- *In those cases where it is a factor, ignoring the difference between Poisson's ratios of model and prototype induces errors that are usually small enough to ignore.*
- *If displacement boundary conditions are imposed, then the modulus of elasticity is an important factor that is easily taken into account.*

4. In the rare case where body forces (e.g. gravity loads) are not zero or a constant, then the Poisson ratio is a factor, but other material coefficients are not.
5. Even for those cases where it is a factor, the error in ignoring the difference between the Poisson ratios for model and prototype is usually much less than 7%, and it can be ignored for most cases.
6. If displacement or strain boundary conditions are specified, which is a relatively rare situation, then, obviously, the stress magnitudes are proportional to the modulus of elasticity, and it is easily taken into account.

The conclusion is that in the great majority of practical applications of photoelasticity, the differences in properties of model and prototype are not relevant as long as both are behaving in a "momentarily linearly elastic" way (see Part 30 of this series). If creep or relaxation is a factor in the problem, then the material similarity conditions are more complex because time must be considered. The rules are also more stringent for non-isotropic materials.

The conclusion is that in most photoelasticity studies, material properties need not be considered when transferring the results of experiment to the prototype.

LOAD MAGNITUDE

This one is easy! Except for certain nonlinear cases, such as near a load application point or where large deflections change the geometry, stress magnitudes are always proportional to the applied load. If P_m is the load applied to the model and σ_m is the resulting stress in the model, and if $P_p$and σ_p are the corresponding quantities for a prototype that is identical to the model in size and shape, then the stress is simply scaled by the load ratio,

$$\sigma_p = \sigma_m \left(\frac{P_p}{P_m}\right) \tag{38.3}$$

The load difference between model and prototype is easily accounted for because stress is proportional to load, other things being equal.

If, as is usually the case, more than one pair of equilibrated loads is applied to the model, then the ratios of like loads between model and prototype must be constant. A similar rule holds for load directions. The model does not accurately represent the prototype unless these conditions are observed.

SIZES AND SHAPES

The rule for shape correspondence between model and prototype is intuitively obvious. Geometric similarity requires that the model be linearly scaled up or down relative to the prototype. That is, the model should look like a photographic enlargement or reduction of the prototype. To do otherwise is to change the problem because the shape is changed.

Geometric similarity requires that the size of the model can be changed relative to the model, but the shape must not be changed. Think of a photographic enlargement or reduction.

The proper way to scale stresses to account for the ratio of model size to specimen size is also easy to work out. Consider again the basic solutions for stresses in structural components, examples of which were presented above. What happens when the size is changed? Take as a first example a tension member. If all dimensions of the specimen are doubled, the cross sectional area is quadrupled, and the tensile stress is reduced by a factor of four. A cantilever beam provides a second guiding example. Increase ALL the dimensions, including the moment arm, by a factor of two, then compute the bending stress at the point whose distance from the neutral axis is also doubled. The stress will again be found to be decreased by a factor of four. These examples lead to the correct idea that stresses are inversely proportional to the square of the size ratio.

Scaling stresses to account for the difference in size between model and prototype is easy to do, because stress is inversely proportional to the square of the magnification.

- *If the model is twice as large as the prototype, then the stresses are reduced by a factor of four.*
- *This conclusion, based on elementary considerations, is supported by strict dimensional analysis.*

One further detail merits attention. In two-dimensional photoelasticity, the ratio of thickness between model and prototype is often made different from the ratio of the lateral dimensions. That is, the thickness need not be scaled by the same factor that is used for the size. This step is taken when, for example, strict proportional scaling would create a specimen that is very thin, might buckle, and would yield very low fringe orders. The modified stress scaling law for this situation is easily worked out by extending the reasoning outlined above.

In photoelasticity, the model-to-prototype thickness ratio can be made different from the size ratio. This practice is often useful.

THE SCALING LAW

Assemble the findings for load and size scaling as developed above to obtain a final solution for the stress in the prototype in terms of the stress in the model and the load, size, and thickness ratios. This result applies to all cases except those exceptions where material properties absolutely must be considered.

$$\sigma_p = \sigma_m \left(\frac{P_p}{P_m}\right)\left(\frac{a_m}{a_p}\right)\left(\frac{d_m}{d_p}\right) \tag{38.4}$$

where: m and p subscripts refer to model and prototype respectively,

$\frac{P_p}{P_m}$ is the load scale factor,

$\frac{a_m}{a_p}$ is the scale factor for size and is the magnification factor,

$\frac{d_m}{d_p}$ is the thickness scale factor.

The analysis outlined above was developed in terms of a single normal stress component for the sake of convenience. Photoelasticity yields directly the principal stress difference or maximum shear stress. The material similarity and scaling rules developed here for normal stress may be applied directly to shear stress. Likewise, they may be used for cases where the photoelastic results are cast in terms of strains.

In words, the scaling law declares that the stress in the prototype equals the stress in the model times (the ratio of prototype load to model load), times (the ratio of model thickness to prototype thickness), and times (the ratio of model size to prototype size).

ADVANCED STUDY

The ways in which material properties affect the relationships between load, body forces, and stresses for shapes having a single boundary (no holes) can be thoroughly analyzed within the limitations of linear behavior by looking at only the compatibility equations of elasticity. One finds that Poisson's ratio enters the formulation for only some body force distributions. For shapes with multiple boundaries, the compatibility equations are necessary but not sufficient. The condition that displacements be single-valued must also be enforced by requiring that the resultant displacement integrated around each and every boundary be zero. In general form, this procedure evolves into "Mitchell's conditions." These equations demonstrate that the Poisson ratio is a factor only when the traction on any one boundary is not self-equilibrated.

The scaling laws for all types of model analysis, including photoelasticity, are developed in a sophisticated way through application of Buckingham's Theorem, otherwise called the Buckingham Pi Theorem. This powerful approach to dimensional analysis and similitude was formalized and published in about 1914 by Edgar Buckingham, a renowned soils physicist, to whom the theorem is usually credited. But, the idea was evidently first utilized by one of our favorite gentleman geniuses, Lord Rayleigh, prior to his description of the method in 1877. Briefly, this weapon, when applied to problems of stress analysis in solid mechanics, requires that the stress be expressed as an unknown function of a set of all the independent dimensionless products (usually ratios) that can be formed from all the variables that might possibly affect the stress. The key point is that the same functional relationship will apply equally to model and prototype, but the relationship need not be known and discovery of it is not necessarily the goal of the analysis. The result is a set of equality conditions for the dimensionless ratios that must be satisfied in order that the dimensionless stress metric for model and prototype be the same. For our problem, the finding is that equation 38.4 is valid and our elementary analysis is supported as long as the size and loads are linearly scaled.

The scaling laws for all types of model analysis, including photoelasticity, are obtained through dimensional analysis based on the Buckingham Pi Theorem, first used by Lord Rayleigh and formalized by Edgar Buckingham. Dimensional analysis is a powerful aid in understanding complex physical phenomena, and it can lead to the maximization of benefit from experiments on problems involving many parameters.

Sometimes the rules for geometric similarity are deliberately violated, the separation of thickness and size scaling factors being an example already mentioned. Another case is when one needs to compensate for large deflections in the model. Dimensional analysis serves as a guide in these matters.

At first immersion, dimensional analysis with the Pi theorem seems to be an arcane subject whose utility is limited and obscure. It is, in fact, a very powerful aid to comprehension of complex relationships in physics and engineering, and it leads to maximum return from simple experiments on problems where a great many parameters might be involved. This subject seems to be largely ignored in our crowded engineering curricula these days, except for some special applications such as the one discussed here. Demands for cost-effective experiments to validate engineering designs and create new knowledge might reverse this unfortunate trend.

WHAT IS NEXT?

Next time we will probably have a go at recording isoclinic fringes and discovering their elementary utilizations. ■

The next article will probably deal with the recording and utilization of isoclinic data.