OPTICAL METHODS Back to Basics *by Gary Cloud*

Optical Methods in Experimental Mechanics
Part 14: Diffraction at a Clear Aperture

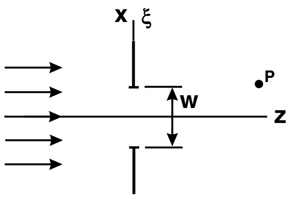
REVIEW AND PURPOSE

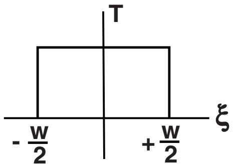
Part 13 showed that an aperture, under certain conditions, creates a visible Fourier transform of the information contained in the aperture.

The purpose here is to predict what should be seen when light is passed through either a long narrow slit or a small circular hole in a plate and to determine whether the predictions are supported by laboratory observation. These two cases are mathematically equivalent and require only a two-dimensional analysis. The circular hole is axisymmetric, so one needs to examine only the medial plane. This problem is not as trivial as it might seem, and the results are important in many applications, including optical imaging, speckle methods, and spatial filtering. The findings, when compared with experience, illustrate some implications of the Fraunhoffer approximations.

PROBLEM AND SOLUTION

For convenience, choose a plane wave front (collimated light) that is illuminating the slit or circular hole at normal incidence as shown in the first sketch below. The width of the slit or the diameter of the hole is w. The problem is to predict the light intensity at some general point P that is a considerable distance away from the aperture.





The second sketch (above) shows the transmittance function for the clear aperture, which, from now on, will be called a hole. The mathematical description of this aperture function, which is a rectangular pulse in space, is,

$$T(\xi) = \mathrm{rect}\left(\frac{\lambda}{w}\right) = \begin{cases} 1 \text{ for } |\xi| \leq \dfrac{w}{2} \\ 0 \text{ for } |\xi| \geq \dfrac{w}{2} \end{cases} \qquad 14.1$$

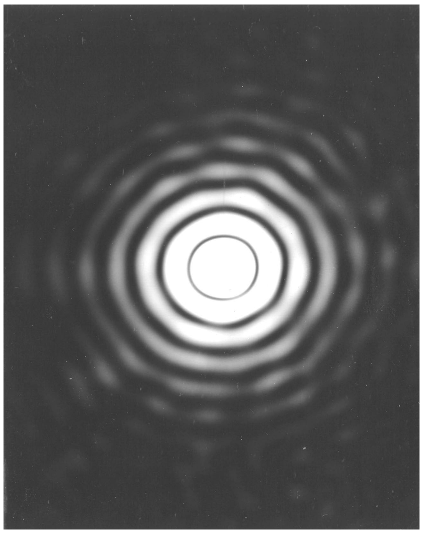
This transmittance function is substituted directly into the Fraunhoffer integral, equation 13.8 of part 13 of this series. Since the illuminating waves are parallel, the source is at infinity and the x_1/z_1 expressions are zero. The transform integral for the scalar complex amplitude at observing point P becomes,

$$U_p = Ce^{\frac{ik}{2}\left(\frac{x_2^2}{z_2}\right)} \int_{-\frac{w}{2}}^{+\frac{w}{2}} e^{-ik\left(\frac{x_2}{z_2}\right)\xi}\, d\xi \qquad 14.2$$

Editor's Note: Optical Methods: Back to Basics, *is organized by ET Senior Technical Editor, Kristin Zimmerman, General Motors, and written by Prof. Gary Cloud of Michigan State University in East Lansing, MI. The series began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless noted otherwise, graphics in this series were created by the author.*

The series author, Prof. Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his book Optical Methods of Engineering Analysis.

If you have any comments or questions about this series, please contact Kristin Zimmerman, Kristin.b.Zimmerman@gm.com.



Diffraction pattern for clear circular aperture, photographed on monochrome film with wide dynamic range so as to show the rings beyond the 7th order. Compare with the digital color picture shown in Part 10. Photo by G. Cloud.

The objectives are to:
- *calculate the diffraction pattern for two mathematically equivalent clear apertures:*
  - *a long narrow slit,*
  - *a circular hole,*
- *compare the predictions with experiment,*
- *explore some implications and applications of the result.*

Collimated light is incident upon a plate containing a small hole or a slit. The complex amplitude at some remote point downstream is sought.

Mathematically, the transmittance function for this aperture is the "rect" or "top-hat."

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The integral is recognized as the Fourier transform of the aperture function and is evaluated as follows, ignoring for the moment the multipliers outside the integral sign,

The complex amplitude at general observing point P is the Fourier transform of the transmittance function times a constant and an obliquity factor. These multipliers can usually be ignored.

$$\boldsymbol{F}\{T(\xi)\} = \left[\frac{e^{-ik\left(\frac{x_2}{z_2}\right)\xi}}{-ik\left(\frac{x_2}{z_2}\right)}\right]_{-\frac{w}{2}}^{+\frac{w}{2}} \tag{14.3}$$

$$= \frac{\sin\left(kw\,\dfrac{x_2}{2z_2}\right)}{k\left(\dfrac{x_2}{2z_2}\right)} \tag{14.4}$$

Recall that $kx_2/2z_2$ is just πf where f is the spatial frequency parameter that is the distance dimension in transform space (see part 13),

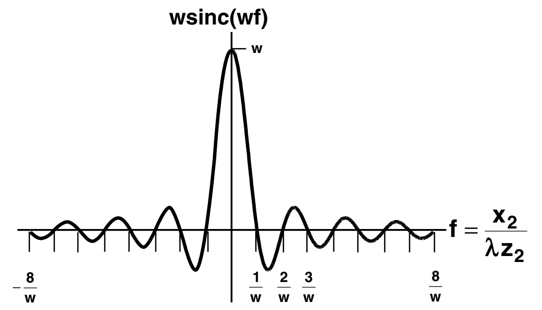
$$f = \frac{x_2}{\lambda\, z_2} \tag{14.5}$$

With this definition in place, the transform can be written in terms of spatial frequency as,

$$\boldsymbol{F}\{T(\xi)\} = \frac{\sin\,\pi\,wf}{\pi\,f} = w\ \mathrm{sinc}(wf) \tag{14.6}$$

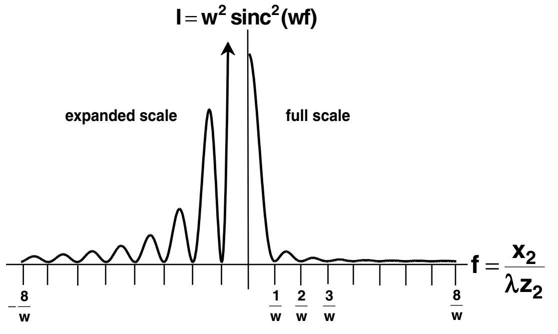
The Fourier transform is of the form (sin ax)/x, which is known as the "sinc function."

A plot of the sinc function appears in the following sketch.



The intensity or irradiance is essentially the square of the sinc function multiplied by the square of the expressions outside the integral sign in equation 14.2. These multipliers can be ignored for development of an understanding of the intensity distribution in a transform plane where z_2 is much larger than x_2. A graph of the square of the sinc function is shown below. Note that the intensity distribution is plotted at an expanded scale in the left-hand portion in order to show clearly the way in which the intensity oscillates and diminishes off-axis.

The intensity distribution is the square of the sinc function.



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The graph shows that, for a long narrow slit, the diffraction pattern will be a system of alternating light and dark bands (fringes) that are parallel to the slit and that rapidly decrease in contrast with increasing distance from the optical axis. For the circular hole aperture, the diffraction pattern will be a central bright patch surrounded by concentric light and dark rings that also rapidly decrease in visibility with increasing distance from the center.

This pattern can be generated easily by passing light from a laser or even a laser pointer through a small pinhole in a sheet of foil. The photograph that appears above is an example of such a pattern that was recorded on monochrome film having a wide exposure latitude so as to preserve the faint bright rings out to and beyond the seventh. Qualitatively at least, this result demonstrates the validity of diffraction theory and the approximations that have been made.

NUMERICAL EXAMPLES AND OBSERVATIONS

This result allows us to understand and predict, among other things, the resolution limit of lenses and the size of laser speckles. As part of the process, one calculates the diameter of the central spot, called the Airy disc, of the diffraction pattern. Examination of the intensity graph suggests that this diameter will be where,

$$2\left(\frac{x_2}{\lambda\, z_2}\right) = 2\left(\frac{1}{w}\right) \qquad 14.7$$

which gives for the diameter of the central bright patch,

$$d = 2x_2 = 2\,\frac{\lambda\, z_2}{w} \qquad 14.8$$

Take, for example, an aperture diameter w of 0.2 mm, use red light at $\lambda = 0.6$ μm, and satisfy the Fraunhoffer restriction by placing the viewing screen quite far from the aperture at $z_2 = 1{,}000$ mm. The diameter of the central disc turns out to be 6 mm. If the aperture diameter is reduced to 0.02 mm, then the central disc diameter expands to 60 mm. The inverse reciprocity between aperture size and expanse of the diffraction pattern is evident. The smaller the aperture, the larger the illuminated area in the diffraction pattern observing plane. This idea deserves to be looked at more closely.

Consider the case where the aperture is vanishingly small. In this case, the aperture function would be the impulse function or Dirac delta. The Fourier transform of the delta function is a constant, meaning that, in this unattainable ideal case, the entire transform plane is illuminated by the central patch of the diffraction pattern.

The other extreme seems troublesome. If the aperture is "large," our theory suggests that the bright patch in the diffraction pattern should be small. That is, the Fourier transform of a constant extending broadly in the aperture plane would approach a delta function in transform space. Stated another way, as the aperture expands, the theory shows that the sinc function becomes narrower.

But, if you project light onto an aperture of, say, 20 mm diameter and if you place a viewing screen at 1000 mm behind the hole, as we did above, you will see on the screen a bright patch that is roughly the size of the aperture. In other words, you see what we usually recognize as the mere shadow of the aperture screen with maybe some visible fuzziness around the edges of a bright spot of 20 mm diameter. Yet, diffraction theory predicts for this case an illuminated central spot having a diameter of only 0.06 mm. What is wrong here?

The problem is that the large-aperture example chosen does not satisfy the Fraunhoffer requirement that the distance to the viewing plane must be much larger than the aperture diameter squared divided by the wavelength (see Part 13, equation 13.7). Specifically, for this case, the viewing distance would need to be much greater than 700 meters in order to see the proper Fraunhoffer pattern

For the circular hole aperture, the diffraction pattern will be a central bright patch surrounded by concentric light and dark rings that rapidly decrease in visibility with increasing distance from the center.

The predictions compare well with laboratory observations.

The central bright patch of the diffraction pattern:
- *is called the Airy disc,*
- *provides a metric for predicting the resolution of optical systems,*
- *is related to the size of laser speckles,*
- *has many other applications.*
- *is a function of:*
  - *aperture size*
  - *distance to the observing plane*
  - *wavelength.*

The relation between aperture size and the breadth of the diffraction pattern is inverse.
- *Small apertures give a large central patch.*
- *Wide apertures give a small central patch.*

If the aperture is large, laboratory observations do not seem to agree with the prediction that the diffraction pattern should be small. We see the "shadow" of the aperture plate on a viewing screen.

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from a 20 mm aperture. This aspect of the Fraunhoffer approximation, which led us to the useful Fourier transform result, is easy to forget; and the forgetting can cause much trouble. It is a problem that is inherent in most treatments of subjects such as moire interferometry and holography that involve diffraction by gratings of broad extent.

The problem is that, for the large aperture, the diffraction pattern must be observed several hundreds of meters away in order to satisfy the Fraunhoffer restrictions.

THE NEXT STEP

The question then arises as to how one might create within the limits of a laboratory of finite size a diffraction pattern from a broad spatial signal such as a photograph. The answer is to use a lens that causes the nearly collimated diffracted beam to converge more quickly. This idea will be studied presently, as will its application in optical spatial filtering. ■

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