OPTICAL METHODS Back to Basics *by Gary Cloud*

Optical Methods in Experimental Mechanics

Part 13: Diffraction Theory, Part III

REVIEW AND PURPOSE

Part 12 developed the Kirchhoff diffraction integral, also called the Fresnel-Kirchhoff formula, giving the complex amplitude anywhere in the region downstream from a diffraction screen that is illuminated by a point source. The Kirchhoff diffraction integral can be evaluated for certain simple apertures and wavefronts, but it proves unwieldy for most practical applications.

This third article on diffraction theory develops some approximations that simplify the integral so it can be used to explain the great majority of diffraction phenomena that are observed and employed in the laboratory. Before taking on that task, an aperture transmittance function is incorporated into the integral equation to facilitate solutions when some intelligence, such as an optical element or a transparency, is included in the aperture.

INCLUSION OF A SIGNAL IN THE APERTURE

A common situation in optical data processing applications is that the diffracting screen usually contains some sort of signal or intelligence, often in the form of a transparency that modifies the amplitude, phase, and/or intensity distribution of the illumination beam as it passes through the aperture. The easiest way to manage this problem is to define an aperture transmittance function $T(\xi,\eta)$, which might be a complex function, so that the complex amplitude U_1 at the aperture is the complex amplitude of the source (see Part 12, Eq. 12.1), evaluated at the aperture, times the transmittance function. From now on, ξ and η represent coordinates in the plane of the aperture and serve as dummy variables for the integration, and x,y, and z remain as global coordinates The derivation of the Fresnel-Kirchhoff equation does not change except that the transmittance function enters the integral as a multiplier to give, for example,

$$U_p = -\frac{Ai}{2\lambda} \iint_{aperture} T(\xi,\eta) \frac{1}{r_1 r_2} e^{ik(r_1+r_2)}(\cos\phi_1 - \cos\phi_2)d\xi d\eta \qquad 13.1$$

GENERAL PLAN OF ATTACK

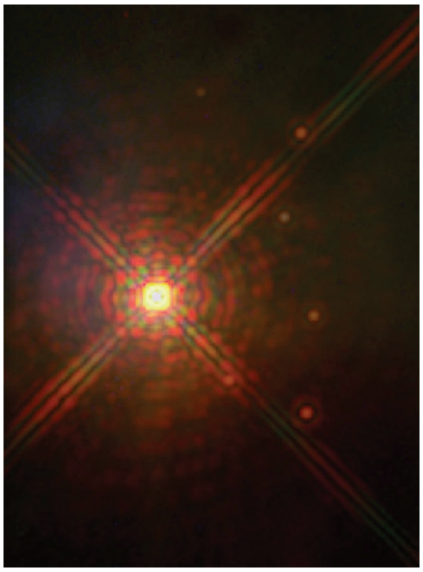
As mentioned, the Fresnel-Kirchhoff integral is difficult to evaluate for even simple apertures and transmittance functions. The reason is that r_1 and r_2 as well as the obliquity factors $\cos\phi_1$ and $\cos\phi_2$ vary as the integration element roams over the aperture, and relationships between these quantities must be established before the integration can be performed. A better approach is to develop some approximations that simplify the integral and facilitate useful solutions for important classes of diffraction problems.

The basic approach to these simplifications is to accept certain limitations on the geometry of the setup, then write corresponding approximations for the position variables. The result is that certain terms in the integral can be dropped entirely and others can be moved outside the integral as constants that do not depend on the location of the integration element.

Editor's Note: Optical Methods: Back to Basics, *is organized by ET Senior Technical Editor, Kristin Zimmerman, General Motors, and written by Prof. Gary Cloud of Michigan State University in East Lansing, MI. The series began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless noted otherwise, graphics in this series were created by the author.*

The series author, Prof. Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his recently published book Optical Methods of Engineering Analysis.

If you have any comments or questions about this series, please contact Kristin Zimmerman, Kristin.b.Zimmerman@gm.com.



Family Portrait*-Hubble Space Telescope NICMOS image of NGC 2264 IRS mother star and baby stars in the Cone Nebula. The rings and spikes emanating from the image form diffraction patterns that demonstrate near-perfect optical performance of the camera. Portion of image no. STScI-PRC1997-16. Image by R. Thompson, M. Rieke, and G. Schneider of University of Arizona and NASA.*

The purposes of this article are to:
- *incorporate a signal that might be included in the aperture,*
- *simplify the diffraction integral for practical applications.*

A transmittance function is included in the diffraction integral.
- *The complex amplitude exiting the aperture is the complex amplitude from the source times the transmittance function.*
- *It might be a complex function.*
- *It can modify the phase or amplitude distributions, or both.*
- *It is defined in local aperture coordinates.*

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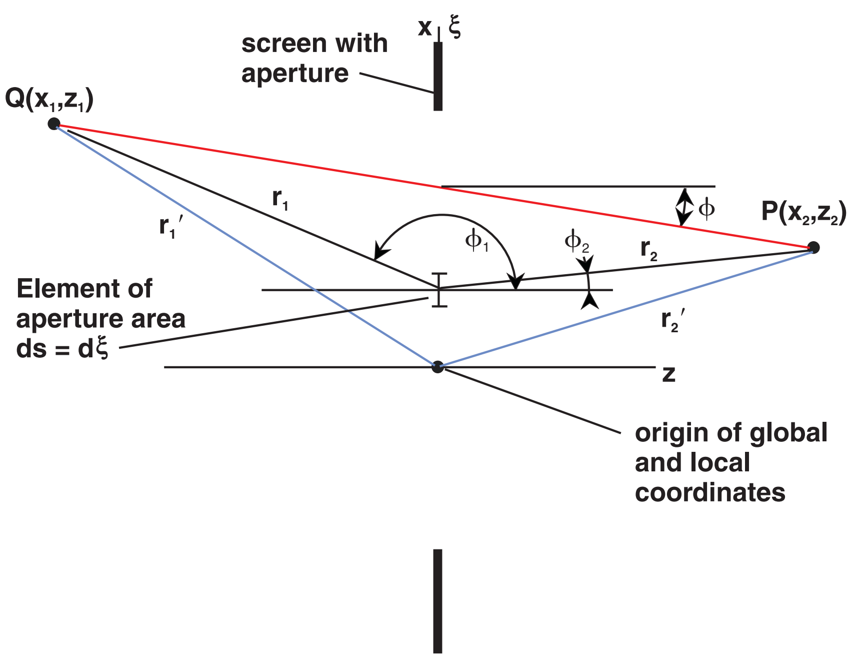
The mathematical development of these approximations is tedious but not profound. Neither does the development involve much physical insight or require physical argument beyond geometry. In order to avoid unprofitable complexity, the problem is reduced to two dimensions; extension to three dimensions being intuitively obvious. Much manipulative detail is omitted, and some sign issues remain unresolved in this compact treatment. Note also that, throughout this series on diffraction, attributions to the leading scholars and the naming of developments after them are not necessarily supported by history or convention.

To simplify the integral, certain limitations on the geometry are accepted.

The first of the simplifying modifications is called the Fresnel approximation. It gives a result that is tractable and quite general in application but still difficult to use. The second, called the Fraunhofer approximation, is more severe in its limitations, less general, but simple and sufficient for most applications.

THE FRESNEL APPROXIMATION

The second figure from Part 12 of this series is now modified by introducing the aperture coordinates mentioned above (actually only one of them, since we are in two dimensions), and incorporating some additional position variables, namely the line QP and the position vectors r_1' and r_2'. The origins for both the global coordinate system (x,z) and the local coordinate ξ are chosen to be at the same point somewhere in the aperture.



Now, assume that the points P and Q are "far" removed from a "small" aperture. If far enough away in comparison with the aperture size, then, for any given location of P and Q,

- The factor $\frac{1}{r_1\, r_2}$ that appears in the diffraction integral can be replaced by the constant $\frac{1}{r_1'\, r_2'}$ and moved outside the integral. The primed distances are to the origin of the global coordinates rather than to the integration element.
- The factor $(\cos\phi_1 - \cos\phi_2)$ in the integral will not change much as the area element $d\xi$ wanders around in the aperture. So, this factor can be replaced by the constant $2\cos\phi$ and moved outside. ϕ is the angle between the normal to the aperture and the line QP. To clarify this point, one can replace the variable angles ϕ_1 and ϕ_2 with the constant angles (for given P and Q) that r_1' and r_2' make with the z-axis, but this measure seems unnecessary.

The Fresnel approximation:

- *assumes that the source and receiving points are "quite far" from the aperture,*
- *assumes that the aperture is "quite small,"*
- *allows some location variables inside the integral to be replaced by constants,*
- *greatly simplifies the integral.*

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- **But**, the $(r_1 + r_2)$ appearing in the exponent inside the integral **cannot** be replaced by $(r_1' + r_2')$ because the exponential oscillates rapidly as the integration element ranges over the aperture.

With these substitutions in place, the diffraction integral becomes much simpler,

$$U_p = -\frac{Ai}{\lambda}\frac{\cos\phi}{r_1' r_2'}\int_{aperture} T(\xi)e^{ik(r_1+r_2)}\,d\xi \qquad 13.2$$

Further important simplification of the exponential in the integrand is achieved by adoption of reasonable approximations to r_1 and r_2. Let us outline the process for only one of the quantities.

$$r_1 = [(x_1 - \xi)^2 + z_1^2]^{1/2} = z_1\left(1 + \frac{x_1^2}{z_1^2} + \frac{\xi^2}{z_1^2} - \frac{2\xi x_1}{z_1^2}\right)^{1/2} \qquad 13.3$$

Invoke the assumption already made that the observing and source points are far enough from a small aperture to claim that $z_1 >> |x_1 - x|$, then use the binomial expansion,

The integrand is further modified by replacing the geometric factors in the exponential with equivalent series expansions.

$$(1 + \varepsilon)^{1/2} = 1 + \frac{\varepsilon}{2} + \cdots \qquad 13.3$$

to expand equation 13.3, then retain only the first terms to obtain,

$$r_1 = z_1 + \frac{x_1^2}{2z_1} + \frac{\xi^2}{2z_1} - \frac{x_1\xi}{z_1} \qquad 13.4$$

Substitute this approximation for r_1 and the equivalent expression for r_2 into the integral of equation 13.2, then move everything not containing the aperture variable ξ outside the integral and rearrange a bit to obtain,

$$U_p = -\frac{Ai}{\lambda}\frac{\cos\phi}{r_1' r_2'}e^{ik(z_1+z_2)}e^{\frac{ik}{2}\left(\frac{x_1^2}{z_1}+\frac{x_2^2}{z_2}\right)}\int_{aperture} T(\xi)e^{ik\frac{\xi^2}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)}e^{-ik\left(\frac{x_1\xi}{z_1}+\frac{x_2\xi}{z_2}\right)}\,d\xi \qquad 13.5$$

This result is the Fresnel approximation to the Kirchhoff diffraction integral. It is not as forbidding as it looks. Most of the expressions outside the integral, including the sign, can be collected into a constant, a measure that is introduced in the next section. This form of the integral is also very powerful, since the simplifying assumptions made so far are not very limiting. It is still more unwieldy than is necessary for most practical applications, so we venture one step further.

The diffraction integral is now much simpler, but it is still too unwieldy for practical applications.

THE FRAUNHOFFER APPROXIMATION

A simpler and more useful result imposes the stringent condition that the exponential

$$e^{\frac{ik\xi^2}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)}$$

be near enough to unity for the full range of ξ encountered so that it may be dropped. A convenient but arbitrary implementation that gives a tidy result is to require that the exponent be much smaller than something already smaller than one for all possible values of ξ.

$$\frac{\pi\xi^2}{\lambda}\left(\frac{1}{z_1}+\frac{1}{z_2}\right) << \frac{\pi}{4} \qquad 13.6$$

At this point, take the maximum size of the aperture to be w, meaning that,

$$\xi_{max} = \frac{w}{2}$$

Substitution into eq. 13.6 gives the Fraunhoffer limitation in terms of easily defined quantities, although the meaning of "much less than" must be explored.

The Fraunhoffer approximation:

- *requires that the source and receiving points are "very far" removed from the aperture,*
- *requires the aperture to be "very small,"*
- *eliminates one of the difficult exponential expressions from the integrand,*
- *places severe physical restrictions on the application of the integral,*
- *reduces the integral to one that is easily evaluated for many applications,*
- *proves very useful, even with its inherent restrictions.*

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$$\frac{1}{z_1} + \frac{1}{z_2} << \frac{\lambda}{w^2} \qquad 13.7$$

Implied is that the source and receiving points must be "very far removed" from a "very small" aperture. This restriction is severe in physical terms, and its implications are not widely acknowledged. But it is satisfied "reasonably well" in many useful applications. More will be said about these limitations presently.

With the restrictions described above in place, and after lumping the geometric constants and rearranging a bit, the diffraction integral of equation 13.5 reduces to the Fraunhoffer approximation,

$$U_p = Ce^{\frac{ik}{2}\left(\frac{x_1^2}{z_1} + \frac{x_2^2}{z_2}\right)} \int_{aperture} T(\xi) e^{-ik\left(\frac{x_1}{z_1} + \frac{x_2}{z_2}\right)\xi} d\xi \qquad 13.8$$

The integral is recognized as a Fourier transform of the aperture transmittance function, and it is often written as,

The diffraction integral becomes the Fourier transform of the aperture function.

$$U_p = Ce^{\frac{ik}{2}\left(\frac{x_1^2}{z_1} + \frac{x_2^2}{z_2}\right)} \boldsymbol{F}\{T(\xi)\}_{f=\frac{1}{\lambda}\left(\frac{x_1}{z_1} + \frac{x_2}{z_2}\right)} \qquad 13.9$$

The leading exponential contains inclination factors that are often ignored. The long subscript appended to the transform symbol specifies the "spatial frequency" or dimension metric in transform space. Physically, the x/z terms are angular deviations from the optical axis (tangent ϕ), the restriction to small angles having already been imposed.

SUMMARY

Diffraction at an aperture is a Fourier transforming process that decomposes optical information (e.g. a picture) into its constituent space-frequency components (e.g. lines per millimeter) that appear at some distance downstream from the aperture. Spatial frequencies in the input plane are translated into illumination at corresponding distances off-axis in the transform pattern. The relative strengths (intensities) of the illumination patches in the transform pattern correspond to the relative weights of the individual spatial frequency components in the original signal.

Diffraction at an aperture decomposes optical information (e.g. a picture) into its constituent space-frequency components (e.g. lines per millimeter).

Distance in a transform plane is proportional to spatial frequency in the aperture signal.

That an aperture is a physical Fourier transforming device, or spectrum analyzer, is thought-provoking, powerful and far-reaching. All of our mathematical lore about Fourier transforms can be called into service. Most of us have been introduced to transforms of time-varying signals. Substitute space (distance) for time, and our learning transfers to the optical domain. Applications include optical data processing, moire interferometry, and holography, to name only a few.

An aperture is a physical Fourier transformer.

WHAT NEXT?

In order to fix and test our findings, the Fraunhoffer approximation of the diffraction integral will be applied to some simple cases that are easily reproduced in the laboratory. ■

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