OPTICAL METHODS Back to Basics *by Gary Cloud*

Optical Methods in Experimental Mechanics
Part 12: Diffraction Theory, Part II

REVIEW AND PURPOSE

Part 11 of this series developed the Helmholtz-Kirchhoff formula that gives the complex amplitude at a point inside a vessel in terms of the complex amplitude distribution on the surface of the vessel.

The objective of this article is to modify and simplify the integral solution so that we approach an answer to the fundamental question, "If light is passed through an aperture, which might contain an optical element, what is the nature of the light field beyond the aperture?"

THE KIRCHHOFF ASSUMPTIONS

Refer to the first illustration and the final equation (eq. 11.10) in the preceeding article. Evaluation of the surface integral requires knowledge of the complex amplitude and its normal gradient on the entire inside surface of the vessel, as well as the location of the point at which the complex amplitude U_P is to be evaluated. One of the many problems involved in this general solution is that reflections inside the vessel should be considered.

Kirchhoff greatly simplified the problem by realizing that most of the vessel is not important. Only the information contained in the illuminated hole in the vessel is critical. So, he eliminated all but the aperture from consideration by assuming that the vessel is very large, very dark, and very nonreflective. Think of a huge box lined with black velvet but with a small hole in one side. The implications are that:

- $U_1 = 0$ and $\dfrac{\partial U_1}{\partial n} = 0$ on the inside surface of the vessel except at the aperture.
- U_1 takes on the same values in the aperture as it would have if the vessel did not exist.

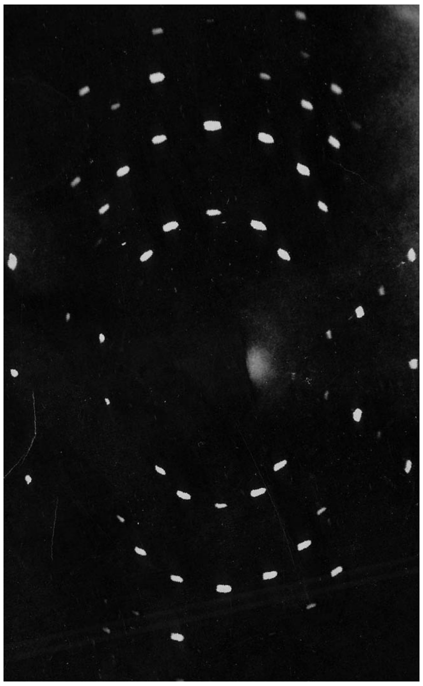
The assumption and its implications are serious and subject to question, and they are aspects of the theory that were refined later by Sommerfield. They do, however, lead us to solutions that have practical value and that can be verified by experiments.

The implications infer that the Helmholtz-Kirchhoff integral over the entire surface is zero except for the part that serves as the aperture. Accordingly, the integration needs to extend only over the domain of the aperture. Therefore, if the nature and position of the radiation source is known relative to the aperture, if the characteristics of the aperture are known, and if the position of the receiving point is known or specified in general form, then the Helmholtz-Kirchhoff integral can be evaluated, at least in principle.

Editor's Note: Optical Methods: Back to Basics, *is organized by ET Senior Technical Editor, Kristin Zimmerman, General Motors, and written by Prof. Gary Cloud of Michigan State University in East Lansing, MI. The series began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless noted otherwise, graphics in this series were created by the author.*

The series author, Prof. Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his recently published book Optical Methods of Engineering Analysis.

If you have any comments or questions about this series, please contact Kristin Zimmerman, Kristin.b.Zimmerman@gm.com.



Transmission Laue X-ray diffraction pattern as used to identify crystal structure and orientation. Courtesy of Dr. K. N. Subramanian, Michigan State University.

The Helmholtz-Kirchhoff integral is to be modified so that it can be applied with ease to useful diffraction problems.

Kirchhoff greatly simplified the problem by assuming that the aperture is a hole in a vessel that is large, dark, and nonreflective, implying that:

- *The complex amplitude and its normal derivative are zero on the inside vessel surface.*
- *There are no reflections or edge effects that modify the complex amplitude at the aperture.*
- *The diffraction integral reduces to a constant = zero everywhere but in the region of the aperture.*
- *The integral needs to be evaluated only over the extent of the aperture.*

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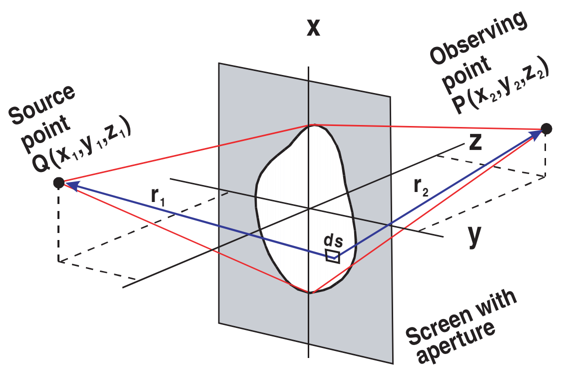
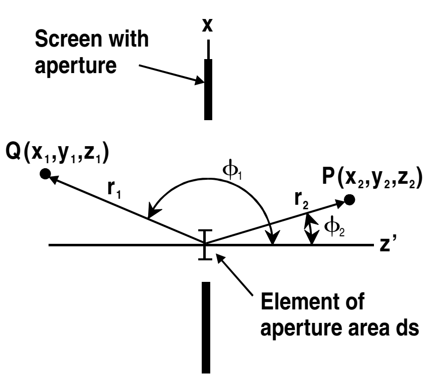
OPTICAL METHODS IN EXPERIMENTAL MECHANICS

THE KIRCHHOFF DIFFRACTION INTEGRAL

The isometric sketch below establishes the geometry of the problem, including the definitions of the position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ for the source point and receiving point *relative to a small element within the aperture*. The two-dimensional illustration clarifies the parameters and defines the two direction angles, ϕ_1 and ϕ_2, taken from a positive z'-axis through the aperture element, for the position vectors.

The positions of the source and receiving points are established with respect to an integration element ds in the aperture.





Suppose that U_1 comes from a point source Q at distance r_1 and inclination ϕ_1 from the *integration element* $ds = dxdy$ *that lies within the aperture*, as shown in the sketches. The aperture is considered to be planar, because eventually it will be taken to be very small relative to the other dimensions involved. Also, the assumption of a point source is not unduly restrictive because an extended source can be viewed as an array of point sources. The complex amplitude falling on the receiving point P at distance r_2 and inclination ϕ_2 from the origin is sought.

A point source outside the vessel is assumed to illuminate the aperture with a spherical wavefront.

The light coming from the point source and incident on the aperture screen can be taken to be a spherical wavefront (see Part 11),

$$U_1 = \frac{Ae^{ikr_1}}{r_1} \tag{12.1}$$

The complex amplitude falling upon a receiving point inside the vessel is sought.

The normal to the aperture is the z'-axis, so,

$$\frac{\partial U_1}{\partial n} = \frac{\partial U_1}{\partial z} = A\frac{e^{ikr_1}}{r_1}\left(ik - \frac{1}{r_1}\right)\cos\phi_1 \tag{12.2}$$

Likewise, the derivative of U_2 that appears in the integral is,

$$\frac{\partial}{\partial n}\left(\frac{e^{ikr_2}}{r_2}\right) = \frac{\partial}{\partial z}\left(\frac{e^{ikr_2}}{r_2}\right) = A\frac{e^{ikr_2}}{r_2}\left(ik - \frac{1}{r_2}\right)\cos\phi_2 \tag{12.3}$$

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A serious but practical assumption is now imposed. Recall that $k = \frac{2\pi}{\lambda}$, meaning it is a large number ($\sim 10^5$ cm^{-1} for visible light). If the source and receiving points are assumed to be somewhat distant (meaning more than a centimeter or so) from the aperture screen, and if the aperture is not too large, then,

$$\frac{1}{r_1} << k \quad \text{and} \quad \frac{1}{r_2} << k \qquad 12.4$$

and so can be dropped.

If the source and receiving points are more than a few centimeters from a relatively small aperture, then certain terms can be dropped from the integral.

The results developed above are substituted into the Helmholtz-Kirchhoff formula to obtain,

$$U_p = -\frac{Aik}{4\pi} \iint\limits_{aperture} \frac{1}{r_1 r_2} e^{ik(r_1+r_2)} (\cos\phi_1 - \cos\phi_2) dxdy \qquad 12.5$$

$$U_p = -\frac{Ai}{2\lambda} \iint\limits_{aperture} \frac{1}{r_1 r_2} e^{ik(r_1+r_2)} (\cos\phi_1 - \cos\phi_2) dxdy \qquad 12.6$$

These equations are slightly different forms of the Kirchhoff diffraction integral, also called the Fresnel-Kirchhoff formula, giving the complex amplitude anywhere in the region downstream from a diffraction screen that is illuminated by a point source.

These simplifications reduce the general diffraction integral to the Kirchhoff integral, also called the Fresnel-Kirchhoff formula, giving the complex amplitude anywhere in the region downstream from a diffraction screen that is illuminated by a point source.

At this point, we see that the diffraction problem reduces to some sort of integral transform, which is encouraging.

THE NEXT STEP

The Kirchhoff diffraction integral can be evaluated for certain simple apertures and wavefronts, but it proves unwieldy for most practical applications. The reason is that the distances r_1 and r_2, as well as the cosines of the direction angles, vary widely as the integration element ds wanders inside the aperture. General relationships between these variables must be found before the integration can be carried out, and this task is usually forbidding.

A better approach is to develop some approximations that simplify the integral and yet supply reasonable solutions for practical diffraction problems. These approximations will be pursued in the third and final article on diffraction theory. At the same time, an aperture transmittance function will be incorporated into the integral equation to facilitate solutions when some intelligence, such as an optical element or a transparency, is included in the aperture. ■

The Kirchhoff diffraction integral:
- *can be evaluated for certain simple cases,*
- *is difficult to evaluate for problems of practical importance,*
- *must be simplified further through development of some approximations,*
- *needs to be modified to account for the case where some intelligence, such as a transparency, is placed in the aperture.*

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