OPTICAL METHODS Back to Basics *by Gary Cloud*

Optical Methods in Experimental Mechanics

Part 11: Diffraction Theory, Part I

REVIEW AND PURPOSE

Part 9 of this series gave some examples of diffraction phenomena drawn from more-or-less everyday experience. Many higher-level scientific examples could be cited, and all of these convince us that understanding the diffraction problem and its solution is important.

Part 9 also stated the fundamental question in its simplest form, “If light is passed through an aperture, which might contain an optical element, what is the nature of the light field beyond the aperture?”

Part 10 developed the concept of “complex amplitude,” which is a descriptor that is convenient for describing the nature and propagation of light waves.

The objective of this article is to derive the Helmholtz-Kirchhoff equation that gives the optical complex amplitude at a point in terms of the complex amplitude that is distributed over a surface around the point. The result is critical to the development and understanding of diffraction theory.

A pedagogical problem is to find an acceptable balance between key concepts, mathematical detail, and physical understanding. More mathematics are presented here than is typical for these articles, because important physical reasoning is tightly woven into the evolution of the theory.

Arcade *Pinhole camera image of shopping arcade taken using a pinhole camera. Note extraordinary depth of field and sharpness obtained by “lensless photography.” Image by Mr. Andrew T. Smith, 2003.**

We seek to relate the optical complex amplitude at a point to the complex amplitude field that surrounds the point.

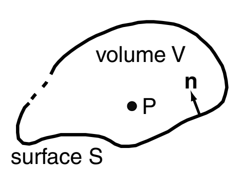
THE HELMHOLTZ-KIRCHHOFF EQUATION

First, the problem is restated in a slightly different way. Given the complex amplitude on the inside surface of a vessel, what is the complex amplitude at some observing point P inside the vessel?

Restate the problem. Given the complex amplitude on the surface of a vessel, what is the complex amplitude at any observation point P *inside the vessel?*

The solution begins in a way that is beguilingly indirect, which is to say it hardly seems related to the problem. Recall Stokes’s theorem for two well-behaved functions U_1 and U_2 that are defined in a volume V surrounded by surface S, as shown in the figure below. Stokes’s theorem relates certain surface and volume integrals containing the functions.

Use Stokes’s theorem, which relates certain surface and volume integrals containing two functions that are defined in the vessel.



$$\iiint_V (U_1\nabla^2 U_2 - U_2\nabla^2 U_1)\, dv = \iint_S (U_1\boldsymbol{\nabla} U_2 - U_2\boldsymbol{\nabla} U_1)\cdot \mathbf{n}\, ds \qquad 11.1$$

where $\mathbf{n}$ is the unit vector normal to the surface, ∇^2 is the Laplace operator, and $\boldsymbol{\nabla}$ is the vector gradient operator.

Editor’s Note: Optical Methods: Back to Basics, *is organized by ET Senior Technical Editor, Kristin Zimmerman, General Motors, and written by Prof. Gary Cloud of Michigan State University in East Lansing, MI. The series began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless noted otherwise, graphics in this series were created by the author.*

The series author, Prof. Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his recently published book Optical Methods of Engineering Analysis.

If you have any comments or questions about this series, please contact Kristin Zimmerman, Kristin.b.Zimmerman@gm.com.

**Image copyright by Mr. Andrew T. Smith of Melbourne, Australia; used with his permission. Exposure 30 seconds at f 199 using 100ASA color negative film in 6cm x 6cm format. In the picture, you can see some ghostly images of people who wandered past during the exposure. A larger version of this picture and additional fine images by Mr. Smith and other artists may be found at the Pinhole Gallery website mentioned at the end of this article.*

Since we are working with light waves, the functions U_1 and U_2 can be taken to be complex amplitudes as were discussed in Part 10 of this series. The implication is that these functions are solutions of the wave equation, so the entire volume integral on the left hand side of equation 11.1 is zero.

The two functions are taken to be complex amplitudes of optical waves.

Further, since

$$\nabla U_i \cdot \mathbf{n} = \frac{\partial U_i}{\partial n} \quad 11.2$$

Because the complex amplitudes are solutions of the wave equation, the entire volume integral portion of Stokes's theorem vanishes.

the surface integral in equation 11.1 reduces to a simple form, and Stokes's equation becomes, for this case,

$$\iint_S \left(U_1 \frac{\partial U_2}{\partial n} - U_2 \frac{\partial U_1}{\partial n} \right) ds = 0 \quad 11.3$$

Imagine that U_1 is a complex amplitude that exists inside the volume. Later, we will find that we need to know its values only on the surface, which is one of the benefits from starting with Stokes's theorem. For example, it might be zero everywhere except at some hole that is illuminated from outside the volume—in other words, an illuminated aperture. If we can figure out what U_2 is everywhere inside the volume in terms of the known surface values of U_1, then the problem is solved. This cannot be done directly in general form, so the unknown complex amplitude inside the surface is taken to be of the form of a spherical wavefront that is centered at the general observation point P. A price paid for this simplification is that the actual complex amplitude at P might be the sum of many such spherical waves, but that possibility causes us no difficulty in the end. The main benefit of this step is that it puts P specifically into the solution. Let r_2 be the distance measured from point P, recall k is the wave number, and the spherical wave complex amplitude is,

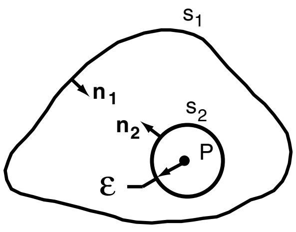
The second complex amplitude is taken to be a spherical wave centered at the observing point P.

$$U_2 = \frac{e^{ikr_2}}{r_2} \quad 11.4$$

This complex amplitude is to be entered into the reduced Stokes's equation and the integral worked out. The trouble is that point P must be seen as another surface, no matter how small, and the integral must include that surface. To accomplish this, surround P with a small sphere of radius ε whose surface is designated S_2 and whose outward normal is $\mathbf{n}_2$, as shown in the following figure. A more conventional view is to claim that P is a singular point and do the same thing, but that rationale seems less than satisfactory here because no volume integral is required. We need to introduce a second surface surrounding P, a measure that causes the complex amplitude at P to appear explicitly in the result.

If the reasoning outlined above seems tedious and unsatisfactory, the problem may be restated as follows. Given the complex amplitude on the surface of the surrounding vessel, predict the complex amplitude on the surface of a small sphere located somewhere inside the vessel. Since the sphere is tiny, the assumption of a spherical wave form is reasonable.

In order to evaluate the surface integral of Stokes's theorem:
- *a second surface is taken to surround* P
- *this surface is a sphere*
- *the radius of the sphere is vanishingly small.*



The integral is now evaluated for both of the surfaces S_1 and S_2 with the adopted expression for U_2 and with the radius of S_2 vanishingly small. Evaluate first the integral over the outside surface, which becomes,

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$$\iint_{S_1} \left[U_1 \frac{\partial}{\partial n_1} \left(\frac{e^{ikr_2}}{r_2} \right) - \left(\frac{e^{ikr_2}}{r_2} \right) \frac{\partial U_1}{\partial n_1} \right] dS_1 \qquad 11.5$$

The integral for the inside sphere is simplified by first realizing that $\frac{\partial}{\partial n_2} = \frac{\partial}{\partial r_2}$, which allows evaluation of the partial derivative of the adopted expression for U_2. This integral becomes,

$$\iint_{S_2} \left[U_1 \left(\frac{ik}{r_2} e^{ikr_2} - \frac{1}{r_2^2} e^{ikr_2} \right) - \frac{e^{ikr_2}}{r_2} \frac{\partial U_1}{\partial n_2} \right] dS_2 \qquad 11.6$$

Now, take $r_2 = \varepsilon$, meaning $dS_2 = \varepsilon^2\, d\Omega$, where $d\Omega$ is the element of solid angle subtended by the element dS_2. The second integral becomes,

$$\int_0^{4\pi} \left[U_1 e^{ik\varepsilon} \left(\frac{ik}{\varepsilon} - \frac{1}{\varepsilon^2} \right) - \frac{e^{ik\varepsilon}}{\varepsilon} \frac{\partial U_1}{\partial n_2} \right] \varepsilon^2\, d\Omega \qquad 11.7$$

The surface integral is evaluated over the two surfaces, using the adopted spherical wave front for the second function.

Since the sphere surrounding P is very small, U_1 cannot vary appreciably on this surface because it is smooth and analytical. So, the entire first expression inside the integral can be taken as near-enough a constant. That portion can be integrated, giving the intermediate result,

$$-4\pi\varepsilon^2 U_1 \frac{e^{ik\varepsilon}}{\varepsilon} \left(ik - \frac{1}{\varepsilon} \right) + \varepsilon \int e^{ik\varepsilon} \frac{\partial U_1}{\partial n_2}\, d\Omega \qquad 11.8$$

It might seem that ground has been lost, but success is near. Although U_1 might be large, its normal derivative is bounded. Therefore, the integral remaining in equation 11.8 is finite, and the entire final expression tends to zero as ε approaches zero. Further, as ε is diminished, the first expression in equation 11.8 becomes,

$$4\pi U_1|_{at\ r_2=0} = 4\pi U_P \qquad 11.9$$

Join this result with the value of the integral over the outside surface (eq. 11.5) to obtain,

$$U_P = \frac{1}{4\pi} \iint_S \left[U_1 \frac{\partial}{\partial n} \left(\frac{e^{ikr_2}}{r_2} \right) - \frac{e^{ikr_2}}{r_2} \frac{\partial U_1}{\partial n} \right] ds \qquad 11.10$$

Since the functions are well-behaved, the integral over the surface of the small sphere surrounding the observing point reduces to a constant times the complex amplitude at the point.

This important result is known as the Helmholtz-Kirchhoff equation. It gives the complex amplitude at a point in terms of the values of the complex amplitude on a surface surrounding the point.

A subtle but interesting point is that we did not evaluate U_2 at P as we might have thought was the objective. A form for U_2 was adopted, and we ended up evaluating U_1 at P. View U_1 and U_2 as different names for the same complex amplitude to clear up this puzzle.

The resulting integral relationship:
- *is known as the Helmholtz-Kirchhoff equation*
- *relates the complex amplitude at an observing point inside a vessel to the values that the complex amplitude has on the surface of the vessel*
- *is difficult to evaluate for practical problems.*

THE NEXT STEPS

The Helmholtz-Kirchhoff equation solves the problem, but it is of limited practical use because of the difficulty of evaluating the integral. The next two articles will press onward with the introduction of some assumptions and approximations that simplify the integral and make it useful.

AUTHOR'S NOTE

Only a few days after the appearance of Part 10, which contained some comments about the camera obscura, I learned from an article in the local newspaper that this phenomenon was known in China around the fifth century B.C.E. I should have known. An interesting web site about the camera obscura is at www.pinhole.org. ■

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