OPTICAL METHODS Back to Basics *by Gary Cloud*

Optical Methods in Experimental Mechanics
Part 10: Complex Amplitude

REVIEW AND PURPOSE

In this series of articles, emphasis is on the physical phenomena that undergird optical methods of measurement, so mathematical manipulation is being held to a minimum. As we begin study of the diffraction problem, however, and then go on to consider more complex optical techniques, we require a way to deal with light waves that is more compact and less unwieldy than the trigonometric approach used so far. This need is satisfied by a descriptor called the "complex amplitude."

WAVE NUMBER

Extract from Part 1 of this series (May/June 2002) the expression that represents the electric vector for a single wave traveling in the z-direction. Change the sign in the argument, which makes no difference to the meaning,

$$\mathbf{E} = \mathbf{A} \cos\left\{\frac{2\pi}{\lambda}[\upsilon t - z]\right\} \qquad 10.1$$

where $\mathbf{A}$ = a vector giving the amplitude and polarization direction of the wave,
λ = the wave length,
υ = the wave velocity.

This equation is easily generalized for a wave propagating along some axis specified by unit vector $\boldsymbol{\alpha}$ having directions l, m, n in Cartesian space. Italic bold type distinguishes the unit vectors.

$$\boldsymbol{\alpha} = l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k} \qquad 10.2$$

The wave traveling in the $\boldsymbol{\alpha}$ direction can be written as,

$$\mathbf{E} = \mathbf{A} \cos\left\{\frac{2\pi}{\lambda}[\upsilon t - (lx + my + nz)]\right\} \qquad 10.3$$

A general position vector locating some arbitrary point in space is,

$$\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad 10.4$$

so the equation for a wave traveling in the specified direction can be written with a dot product term that contains wavelength and direction,

$$\mathbf{E} = \mathbf{A} \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \qquad 10.5$$

where $\mathbf{k}$ = vector wave number = $\mathrm{k}\boldsymbol{\alpha} = \dfrac{2\pi\boldsymbol{\alpha}}{\lambda}$

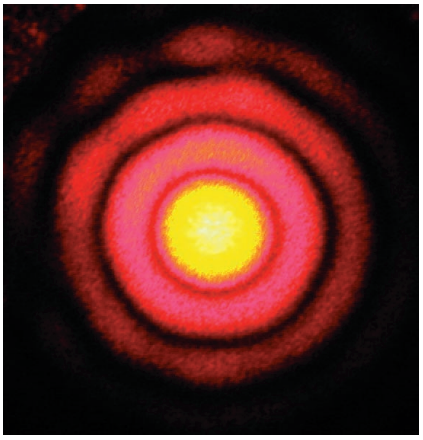
ω = angular frequency of radiation = $\dfrac{2\pi\upsilon}{\lambda} = 2\pi\nu$

ν = optical frequency, Hz

Editor's Note: Optical Methods: Back to Basics, *is organized by ET Senior Technical Editor, Kristin Zimmerman, General Motors, and written by Prof. Gary Cloud of Michigan State University in East Lansing, MI. The series began by introducing the nature and description of light and will evolve, with each issue, into topics ranging from diffraction through phase shifting interferometries. The intent is to keep the series educationally focused by coupling text with illustrative photos and diagrams that can be used by practitioners in the classroom, as well as in industry. Unless noted otherwise, graphics in this series were created by the author.*

The series author, Prof. Gary Cloud (SEM Fellow), is internationally known for his work in optical measurement methods and for his recently published book Optical Methods of Engineering Analysis.

If you have any comments or questions about this series, please contact Kristin Zimmerman, Kristin.b.Zimmerman@gm.com.



Diffraction pattern created by passing laser light through a pinhole. HeNe laser. Central portion overexposed to show the first few off-axis rings. Digital photo by Gary Cloud, Dec. 2003.

We require a representation for light waves that is easier to use than the cosine-wave form for the electric vector.

The cosine wave is first generalized to describe waves traveling in any direction.

The cosine wave is then written in a form that contains:
- *the vector wave number, which specifies:*
  - *propagation direction*
  - *wave length*
- *the angular frequency of the radiation, which is related to:*
  - *wavelength*
  - *wave velocity*

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OPTICAL METHODS IN EXPERIMENTAL MECHANICS

Caution: do not confuse the vector wave number **k** or its scalar counterpart k with the unit vector ***k***.

SCALAR COMPLEX AMPLITUDE

The representation of the wave in equation 10.5 is physically meaningful and sufficient for many optical calculations. But, it tends to be cumbersome when describing the propagation and interactions of waves in the more complicated optical systems, such as those that are used for holographic interferometry. A complex number representation is better, although more difficult to interpret.

Use the identity $e^{i\theta} = \cos\theta + i\sin\theta$ to convert the electric vector to the equivalent exponential form. Only the cosine part is needed.

$$\mathbf{E} = \text{Re}\,\mathbf{A}e^{i[\omega t - \mathbf{k}\cdot\mathbf{r}]} \qquad 10.6$$

Where "Re" means "real part of" and i is $\sqrt{-1}$, often represented by j. Henceforth, the "Re" will be understood to be applicable wherever it is appropriate, so it is dropped from the equations.

The cosine wave is converted to exponential form, and:
- *a phase term is introduced*
  - *the phase term is the magnitude of the vector wave number times the PLD*
- *the part containing the optical oscillation frequency is dropped, since these frequencies are too large to be observed*
- *The polarization specification is also dropped.*

Now, it is a simple matter to include a phase-angle term ϕ, which is how we get the entire path length (PL) or the path length difference (PLD) into the picture. The phase angle could have been introduced into the cosine wave function above, of course. Multiply the physical PLD by the wave number to convert it to phase angle.

$$\phi = \frac{2\pi}{\lambda}(\text{PLD}) = \text{k}(\text{PLD}) \qquad 10.7$$

The expression for the electric vector with the PLD contained becomes,

$$\mathbf{E} = \mathbf{A}e^{i[\omega t - \mathbf{k}\cdot\mathbf{r} + \phi]} \qquad 10.8$$

This representation of the general wave can also be written as a product,

$$\mathbf{E} = \mathbf{A}e^{i\omega t}e^{i\phi}e^{-i\mathbf{k}\cdot\mathbf{r}} \qquad 10.9$$

The first exponential in the above equation represents the oscillation at optical frequencies. We have no way of tracking signals at these high frequencies, so it makes sense to leave that term out. The amplitude data, the space variables, and the phase are the only quantities of interest. Also, for interference to take place, the polarization direction in a setup must be uniform, so we usually drop the vector designations on the electric vector **E** and the amplitude vector **A**. What is left is a much simplified representation of the wave that contains all the important information. It is called the scalar complex amplitude, U.

$$U = Ae^{i\phi}e^{-i\mathbf{k}\cdot\mathbf{r}} = Ae^{i(\phi - \mathbf{k}\cdot\mathbf{r})} \qquad 10.10$$

What is left is the "complex amplitude" that contains as a function of position:
- *amplitude of the wave*
- *phase*
- *wavelength*
- *propagation direction*

In many instances, the amplitude and the phase angle are functions of location in the coordinate system, so we would write $A(x,y,z)$ and so on. As an example, a bundle of waves traveling in the y direction and having amplitude and phase varying with position could be expressed as,

$$U = A(x,y,z)e^{i[\phi(x,y,z) - \text{k}y]} \qquad 10.11$$

Note that there is nothing implicit here that indicates the breadth of the wave bundle. The same equation serves for just one wave or for a broad beam of many waves.

A detail that often proves useful in optics calculations is that twice the real part of a complex number is the sum of the complex number and its complex conjugate. The factor of 2 is often absorbed in the other constants and does not usually appear explicitly. An asterisk is used here to indicate complex conjugate.

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$$U + U^* = 2\,\text{Re}(U) \qquad 10.12$$

INTENSITY OR IRRADIANCE

We learned in Part 2 that the essence of interferometry is in determining invisible phase quantities by converting them to palpable intensity or irradiance. The irradiance is defined, somewhat arbitrarily, as twice the average over many optical oscillations of the square of the magnitude of the electric vector. That is, where T is several optical oscillation periods and where r represents the coordinates of a point in space where the intensity is to be determined,

$$I(r) = \frac{2}{T}\int_0^T E^2(r,t)dt \qquad 10.13$$

Taking the trigonometric form for a simple harmonic wave for example, and letting $\langle\ \rangle$ represent the average over several oscillations,

$$I(r) = 2A^2(r)\langle\cos^2(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi(r))\rangle \qquad 10.14$$

The time average over many periods of the cosine-squared function is just 1/2, so,

$$I(r) = A^2(r) \qquad 10.15$$

Which is to say the irradiance at a point turns out to be the square of the amplitude at that point.

To establish the relationship between intensity and complex amplitude, write the electric vector in terms of complex amplitude and its complex conjugate as mentioned in equation 10.12.

$$E(r,t) = \frac{1}{2}(Ue^{i\omega t} + U^*e^{-i\omega t}) \qquad 10.16$$

Put this identity into the definition of intensity (equation 10.13 or 10.14), leave an extra 1/2 out as is usual, and obtain,

$$I(r) = \langle U^2e^{i2\omega t} + U^{*2}e^{-i2\omega t} + UU^*\rangle \qquad 10.17$$

Recognize that,

$$\langle e^{\pm i2\omega t}\rangle = 0 \qquad 10.18$$

and the intensity reduces to,

$$I(r) = UU^* = |U|^2 \qquad 10.19$$

This result shows that the intensity of a wave at a point is the square of the modulus or amplitude of the complex amplitude at that point, which agrees with the result of equation 10.15.

Interference and diffraction calculations require tracking the complex amplitude through the system and then determining the intensity distribution in the field by multiplying the resulting complex amplitude by its complex conjugate. Often, the result is transformed back into trigonometric form to make it easier to interpret. ■

Intensity or irradiance is:
- *defined as twice the long-time average of the square of the amplitude*
- *the quantity of interest since it is what we can measure*

Intensity at a point in the optical field is found to equal the square of the local amplitude.

Intensity is also found to equal:
- *the local complex amplitude times its complex conjugate*
- *the square of the modulus of the complex amplitude.*

Optical calculations involve:
- *determining the complex amplitude field as waves interact with optical components and other waves*
- *converting the complex amplitude to intensity distribution*

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